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A real-space renormalisation group approach to electrical and noise properties of percolation clusters

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Abstract. The Migdal-Kadanoff renormalisation scheme, which becomes exact on hierarchical lattices, is used to investigate electrical properties of 2D and 3D percolation clusters. We study in detail the frequency dependence of the conductivity and the loss angle, and the amplification of resistance (flicker) noise. This type of noise exhibits critical amplification, with its own exponents X and Y, which are related to the exponent b recently defined by Rammal *et al.*

1. Introduction

The properties of random resistor networks near their percolation threshold have motivated a lot of recent work. The static (DC) conductivity of such systems has been studied by various methods: transfer matrices (Derrida *et al* 1982, 1983, Herrmann *et al* 1984), real-space renormalisation group (Stinchcombe and Watson 1976, Bernasconi 1978, Wilkinson *et al* 1983), series expansion (Fish and Harris 1978), Monte Carlo method (Pandey and Stauffer 1983) and $\varepsilon = 6 - D$ expansion (Harris *et al* 1975). The essential result is the occurrence of two critical exponents, t and s (which are not related to the static exponents like $\alpha, \beta, \gamma, \nu$) characterising respectively the conductivity of a conductor (fraction p)-insulator (fraction $(1-p)$) mixture: $\Sigma \sim (p - p_c)^t$ for $p > p_c$, and that of a superconductor-normal conductor mixture: $\Sigma \sim (p_c - p)^{-s}$ for $p < p_c$.

More generally, the conductivity $\Sigma(\sigma_1, \sigma_2, p)$ of a random mixture of two kinds of elements with conductances σ_1 (fraction p) and σ_2 (fraction $(1-p)$) exhibits the following scaling behaviour:

$$\Sigma(\sigma_1, \sigma_2, p) = \sigma_1 |p - p_c|^t \varphi_\pm[(\sigma_2/\sigma_1)|p - p_c|^{-s-t}] \tag{1.1}$$

when $(p - p_c)$ and σ_2/σ_1 are simultaneously small. The subscript $\pm$ corresponds to $p > p_c$ and $p < p_c$ respectively. If σ_2 goes to zero at fixed σ_1 for $p > p_c$, or if σ_1 goes to infinity at fixed σ_2 for $p < p_c$, equation (1.1) reproduces the above mentioned behaviour of Σ, and hence $\varphi_+(0)$ is finite, while $\varphi_-(x) \sim x$ for small x.

When x is large, φ_+ and φ_- both behave like Kx^u, such that:

$$\Sigma(\sigma_1, \sigma_2, p_c) \sim K \sigma_1^{1-u} \sigma_2^u \tag{1.2}$$

for $\sigma_2 \ll \sigma_1$. The universal exponents s, t, u are related through

$$u = t/(s+t). \tag{1.3}$$

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The scaling laws (1.1)-(1.3) have been proposed in analogy with the equation of state of a ferromagnet near its critical temperature (Webman *et al* 1975, Efros and Shklovskii 1976, Straley 1976, 1977), and derived by field theoretical arguments near $D_c = 6$ (Stephen 1978). The functions $\varphi_\pm$ are universal.

Consider now the AC conductivity of a mixture of resistors ($\sigma_1 = \sigma_0$) and perfect capacitors ($\sigma_2 = i\omega C_0$). Its low-frequency behaviour is given by the analytic continuation of equation (1.1) to an imaginary argument: $\sigma_2/\sigma_1 = i\omega/\omega_0$, where $\omega_0 = \sigma_0/C_0$ is the microscopic frequency scale. This has a well known consequence: the static dielectric constant

$$\varepsilon = \lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} \Sigma(\omega) \tag{1.4}$$

diverges as

$$\varepsilon \sim (p_c - p)^{-s} \tag{1.5}$$

for $p \to p_c^-$ (Efros and Shklovskii 1976, Stroud and Bergman 1982, Wilkinson *et al* 1983).

It has recently been realised that percolation clusters have other amazing electrical properties, like the frequency dependence of the loss angle, which has been measured (Laugier 1982) in metallic and glass microbeads mixtures, as well as noise amplification (Rammal *et al* 1985, Rammal 1984).

An interesting approach to this type of properties is the study of simple analytically tractable models where percolation clusters are replaced by inhomogeneous deterministic fractals (Clerc *et al* 1984, 1985, De Arcangelis *et al* 1984).

The aim of this article is to treat the same kind of problems by the Migdal-Kadanoff approximation (Migdal 1976, Kadanoff 1976). This real-space renormalisation scheme, which is approximate for models on regular lattices, becomes *exact* (at least for non-random models) on recursively built objects called hierarchical lattices (Berker and Ostlund 1979).

The paper is organised as follows. In § 2, we recall the basic properties of two simple hierarchical lattices having dimension 2 and 3 respectively, and solve the static percolation problem on these systems. The 2D lattice is self-dual. Section 3 is devoted to the above mentioned conductivity $\Sigma(\sigma_1, \sigma_2, p)$. We project the renormalisation transformation onto a finite number of recursion relations. The loss angle is studied in great detail. In § 4, we apply the same method to the problem of amplification of flicker noise. This noise amplification obeys scaling laws with its own critical exponents X and Y. A comparison with recent results (Rammal 1984, Rammal *et al* 1985) is presented in § 5, as well as some conclusive remarks.

2. Hierarchical lattices and percolation

Let us first recall briefly the definition of the diamond hierarchical lattice. It is built recursively, by replacing at each step each bond by four new ones, as indicated in figure 1. At the Nth generation, the volume (number of bonds) is $V = 4^N$, while the length between the roots A and B is $L = 2^N$. The diamond lattice has therefore a dimension D (such that $V \sim L^D$) equal to 2.

Similar lattices with higher dimension can be obtained by replacing each bond by a more complicated cell. The simplest lattice having $D = 3$ corresponds to a cell containing four branches of two bonds each, as indicated in figure 2 ($V = 8^N$; $L = 2^N$).

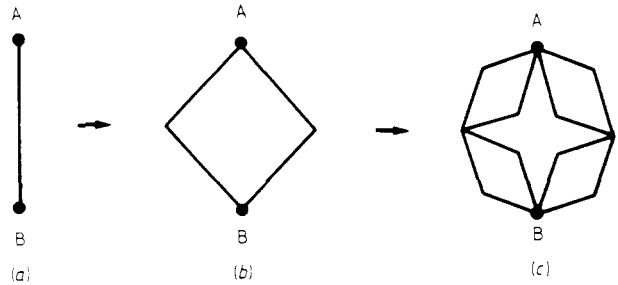

Figure 1. First three steps of the construction of the 2D (diamond) hierarchical lattice.

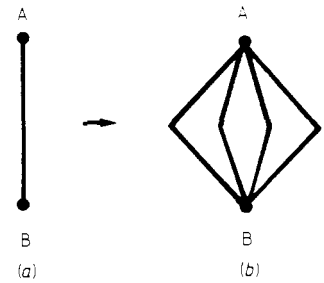

Figure 2. First two steps of the construction of the 3D hierarchical lattice.

We shall consider throughout the following only the lattices of figures 1 and 2.

The bond percolation problem is very easy to solve on these hierarchical lattices: one can find an *exact* renormalisation transform T such that the lattice at the Nth generation with a probability p for a bond to be occupied is strictly equivalent to a lattice at the $(N-1)$th generation with a *renormalised* probability $T(p)$. The quantity $T(p)$ can be viewed as the probability for A and B to be joined at generation $N=1$ (see figures 1 and 2). It is therefore easy to show that the mapping T reads:

$$T(p) = 1-(1-p^2)^2 \qquad (D=2) \tag{2.1a}$$

$$T(p) = 1-(1-p^2)^4 \qquad (D=3). \tag{2.1b}$$

Another way of deriving these expressions is to use the well known Kasteleyn-Fortuin (1969) equivalence between bond percolation and the q-states Potts model in the $q \to 1$ limit. The solution of the Potts model on the hierarchical lattices under consideration (Derrida *et al* 1983, 1984, Itzykson and Luck 1983) leads to equation (2.1) for $q=1$. The rest of this section reports briefly in the percolation language the results obtained in those papers on the Potts model.

In the physical region ($0 \leqslant p \leqslant 1$), the mapping T has two superstable fixed points at $p=1$ and $p=0$, describing percolating and non-percolating pure phases respectively, and one unstable fixed point p_c such that:

$$\begin{aligned} &T(p_c) = p_c \\ &\mathrm{d}T(p_c)/\mathrm{d}p = \mu > 1 \end{aligned} \tag{2.2}$$

which corresponds to the percolation threshold. The critical exponent ν is also easily

obtained as follows. Since the correlation length $\xi(p)$ obeys the functional equation:

$$\xi(p) = 2\xi(T(p)) \tag{2.3}$$

it can be shown (Derrida *et al* 1984) that it behaves for $p \to p_c^{\pm}$ as:

$$\xi(p) = |p - p_c|^{-\nu} P_{\pm}(\ln|p - p_c|/\ln \mu) \tag{2.4}$$

where the critical exponent ν reads:

$$\nu = \ln 2/\ln \mu \tag{2.5}$$

and where $P_{\pm}$ are periodic functions of their argument, with period one. These periodic critical amplitudes usually exhibit *very small* oscillations (Derrida *et al* 1984, Itzykson and Luck 1983) (in other words $P_{\pm}$ are very close to being constants). We shall systematically forget about this oscillatory behaviour in the following, and consider equation (2.4) and similar ones as pure power laws.

The numerical values of p_c and ν read:

$$D = 2 \qquad \begin{aligned} p_c &= \tfrac{1}{2}(\sqrt{5} - 1) = 0.618\,033 \\ \nu &= 1.635\,279 \end{aligned} \tag{2.6a}$$

$$D = 3 \qquad \begin{aligned} p_c &= 0.281\,837 \\ \nu &= 1.227\,411. \end{aligned} \tag{2.6b}$$

The 2D diamond lattice has the interesting property of being *self-dual*, just as the regular square lattice. Figure 3 shows that the dual lattice of the diamond at generation N is nothing other than two diamond lattices at generation $(N-1)$ with double bonds and sharing one root point. It has been shown by Itzykson and Luck (1983) that this (geometrical) duality implies that the renormalisation transform T is the square of a duality mapping D:

$$T = D \circ D \tag{2.7}$$

with

$$D(p) = 1 - p^2. \tag{2.8}$$

The transformation D is different from the duality δ on a regular square lattice:

$$\delta(p) = 1 - p. \tag{2.9}$$

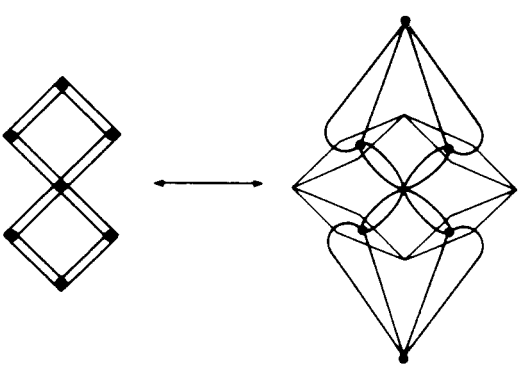

Figure 3. The 2D diamond lattice is geometrically self-dual.

In particular the self duality points are $\frac{1}{2}(\sqrt{5}-1)$ for D, and $\frac{1}{2}$ for δ.

Hierarchical lattices have the considerable advantage that the static percolation properties, in particular p_c and ν, are exactly known. These values are of course different from those of regular (square, cubic) lattices:

$$D=2 \qquad \begin{matrix} p_c=\frac{1}{2}\,(\text{exact}) \\ \nu=\frac{4}{3}\,(\text{den Nijs conjecture (1979)}) \end{matrix} \tag{2.10a}$$

$$D=3 \qquad \begin{matrix} p_c=0.2492\pm 0.0002 \\ \nu=0.88\pm 0.01 \end{matrix} \tag{2.10b}$$

(Heermann and Stauffer 1981, Wilke 1983), but the global picture is correct: see in particular the decrease of p_c and ν between the 2D and the 3D case. Hence we have a sensible framework to study random resistor systems in the following sections.

3. The conductance of a random mixture

3.1. Generalities

The problem we pose ourselves in this section is the following: determine the macroscopic conductivity Σ of our 2D or 3D hierarchical lattice, assuming that each bond has a conductance which is randomly chosen between the two values: σ_1 (probability p) and σ_2 (probability $(1-p)$). We consider the general case of complex conductances (admittances), with emphasis on the resistor-capacitor mixture mentioned in the introduction ($\sigma_1=\sigma_0$, $\sigma_2=i\omega C_0$).

Consider first the 2D lattice. If $\sigma_1 \ldots \sigma_4$ denote the conductances of the four bonds appearing at generation $N=1$, then the conductance σ_R of the whole $N=1$ lattice reads:

$$\sigma_R=\frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}+\frac{\sigma_3\sigma_4}{\sigma_3+\sigma_4}. \tag{3.1}$$

Therefore, if $P(\sigma)$ denotes the probability density of the conductance of one bond, the *renormalised* probability density $P_R(\sigma_R)$ is given by:

$$P_R(\sigma_R)=\int \ldots \int P(\sigma_1)\,d\sigma_1 \ldots P(\sigma_4)\,d\sigma_4 \delta\left(\sigma_R-\frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}-\frac{\sigma_3\sigma_4}{\sigma_3+\sigma_4}\right). \tag{3.2}$$

We have therefore to study the mapping $\mathcal{T}$

$$\mathcal{T}: P(\sigma)\mapsto P_R(\sigma_R)$$

in the functional space of all probability measures. This difficult task, which is characteristic of real-space renormalisation group treatments of disordered systems, can only be achieved through numerical techniques, or analytically in some limiting cases (Derrida and Gardner 1984), or else by the replacement of the infinite-dimensional transform by a finite number of judiciously chosen recursion relations (see Stinchcombe 1983 for a review). We shall use this method in the following, and discuss alternative approaches in § 6.

3.2. Recursion relations and critical exponents

Let us first remark that equation (3.1) can be decomposed into the following steps:

(i) evaluation of the impedances z_α, z_β of the two branches between A and B according to:

$$z_\alpha = z_1 + z_2 \qquad z_\beta = z_3 + z_4 \tag{3.3a}$$

(impedance addition for elements in series)

(ii) evaluation of σ_R in terms of $\sigma_\alpha = z_\alpha^{-1}$, $\sigma_\beta = z_\beta^{-1}$ according to:

$$\sigma_R = \sigma_\alpha + \sigma_\beta \tag{3.3b}$$

(conductance addition for elements in parallel).

We have therefore to investigate only the probability distribution of *a sum*. Since the original distribution of each bond reads:

$$Q(z) = p\delta(z - Z_1) + (1-p)\delta(z - Z_2) \tag{3.4}$$

where we can assume $|Z_1| < |Z_2|$, (otherwise exchange Z_1 and Z_2, p and $(1-p)$), the variables z_α, z_β are distributed with:

$$Q * Q(z) = p^2\delta(z - 2Z_1) + 2p(1-p)\delta(z - Z_1 - Z_2) + (1-p)^2\delta(z - 2Z_2). \tag{3.5}$$

Let us look for an 'optimal' binary distribution:

$$Q'(z) = p'\delta(z - Z_1') + (1-p')\delta(z - Z_2') \tag{3.6}$$

which is as close to $Q * Q$ as possible (once some closeness criteria have been given). Since Q' contains five real parameters (Z_1', Z_2' are complex), we can fix five real conditions: the equalities

$$\begin{aligned} \langle z \rangle_{Q*Q} &= \langle z \rangle_{Q'} \\ \langle z^{-1} \rangle_{Q*Q} &= \langle z^{-1} \rangle_{Q'} \end{aligned} \tag{3.7}$$

provide us with four of them. The choice of a fifth relation has to be guided by the fact that the interesting region for critical behaviour is $|\sigma_1| \gg |\sigma_2|$ or $|Z_1| \ll |Z_2|$. In that limit, the two values $2Z_2$ and $Z_1 + Z_2$ have the same order of magnitude, while $2Z_1$ is much smaller. We find it therefore reasonable to impose

$$p' = p^2 \tag{3.8}$$

(we have assumed $|Z_1| < |Z_2|$).

This equality completely determines Q'. The impedances Z_1', Z_2' read:

$$Z_1' = \frac{ab - 1 + 2p^2 - \Delta^{1/2}}{2bp^2} \tag{3.9a}$$

$$Z_2' = \frac{ab + 1 - 2p^2 + \Delta^{1/2}}{2b(1-p^2)} \tag{3.9b}$$

with the notation

$$a = 2pZ_1 + 2(1-p)Z_2 \tag{3.9c}$$

$$b = \frac{p^2}{2Z_1} + \frac{2p(1-p)}{Z_1 + Z_2} + \frac{(1-p)^2}{2Z_2} \tag{3.9d}$$

$$\Delta = (ab-1)[ab - (2p^2-1)^2] \tag{3.9e}$$

and with the square-root branch analytically continued from $Z_1' = 0$ at $Z_1 = 0$. When

$|Z_1| > |Z_2|$, we just exchange Z_1 and Z_2, p and $(1-p)$, before using equations (3.8)-(3.9). The way we have truncated equation (3.2) to obtain equations (3.8) and (3.9) cannot be *a priori* justified, since our choice is essentially motivated by physical arguments. Let us mention that we have tried numerous other truncation schemes, e.g. by requiring that the average of a third quantity $\varphi(z)$, besides z and z^{-1}, is preserved (see equation (3.7)):

$$\langle \varphi(z) \rangle_{Q*Q} = \langle \varphi(z) \rangle_{Q'}.$$

Quantities like $\varphi(z) = z^2$ violate the duality symmetry; the quantity $\varphi(z) = \mathrm{Re} \ln z$, which respects this symmetry, and which is equivalent to our final choice (3.8) for $|Z_1| \ll |Z_2|$, violates elementary monotonicity properties when iterated.

If we apply our recursion relations (3.8)-(3.9) to bond impedance addition (3.3*a*) and then to branch conductance addition (3.3*b*), we get an approximate renormalisation group transform $\bar{\mathcal{T}}$, which can be seen as being the projection of $\mathcal{T}$ onto some five-dimensional (real) subspace. $\bar{\mathcal{T}}$ has the following drawback: it is not continuous at points such that $|Z_1| = |Z_2|$, where the value of p' jumps from p^2 (3.8) for $|Z_1| < |Z_2|$ to $1-(1-p)^2 = 2p - p^2$ for $|Z_1| > |Z_2|$ (exchange the roles of Z_1 and Z_2, p and $(1-p)$). We shall see hereafter that this discontinuity is harmless in practical situations.

Although we have considered up to now the 2D lattice, the transform $\bar{\mathcal{T}}$ has only to be slightly modified to treat the 3D case, where equation (3.3*b*) reads:

$$\sigma_R = (\sigma_\alpha + \sigma_\beta) + (\sigma_\gamma + \sigma_\delta) \tag{3.10}$$

(α, β, γ, δ denote the four branches of a 3D cell). In order to determine the distribution of σ_R, we just apply *twice* our convolution algorithm, as suggested by the parentheses in equation (3.10).

The action of $\bar{\mathcal{T}}$ on p is independent of Z_1, Z_2 (provided $|Z_1| < |Z_2|$). Moreover, it is soon realised that it reproduces exactly the static mappings T defined in § 2. Since the sequence $T^n(p)$ converges towards 0 ($p < p_c$) or 1 ($p > p_c$), the sequence $\bar{\mathcal{T}}^n(\sigma_1, \sigma_2, p)$ is asymptotic to a *non-random* conductance $\sigma^{(n)}$ which is related to the macroscopic *conductivity* $\Sigma(\sigma_1, \sigma_2, p)$ as follows:

$$\Sigma(\sigma_1, \sigma_2, p) = \sigma^{(n)} L^{2-D} \tag{3.11}$$

where $L = 2^n$ is the length of the lattice. This classical definition of conductivity as a function of conductance and geometrical characteristics ensures that Σ is finite for any $p \neq p_c$, and that $\Sigma = \sigma_0$ for a pure medium ($\sigma_1 = \sigma_2 = \sigma_0$).

Consider first the conductor-insulator mixture ($\sigma_1 = \sigma_0$, $\sigma_2 = 0$) which is also the zero-frequency limit of the resistor-capacitor mixture. In this case, the action of $\bar{\mathcal{T}}$ is very simple:

$$\bar{\mathcal{T}} \qquad \begin{aligned} p &\mapsto T(p) \\ \sigma_0 &\mapsto 2^{D-2} f_0(p) \sigma_0 \end{aligned} \tag{3.12}$$

where $T(p)$ is as in (2.1) and

$$f_0(p) = 1/(2-p^2) \qquad (D=2) \tag{3.13a}$$

$$f_0(p) = 1/(2-p^2)[1+(1-p^2)^2] \qquad (D=3) \tag{3.13b}$$

and therefore the conductivity $\Sigma(\sigma_0, 0, p)$ reads:

$$\Sigma(\sigma_0, 0, p) = \sigma_0 \prod_{n \geqslant 0} f_0(T^n(p)). \tag{3.14}$$

This expression vanishes as it should for $p \leqslant p_c$, reaches the value σ_0 for $p = 1$, and behaves as $(p - p_c)^t$ for $p \to p_c^+$, where the critical exponent t is given by:

$$t = -\ln f_0(p_c)/\ln \mu \tag{3.15}$$

or numerically

$$t = 1.135\,279 \qquad (D = 2) \tag{3.16a}$$

$$t = 2.242\,559 \qquad (D = 3). \tag{3.16b}$$

The superconductor problem ($\sigma_1 = \infty$, $\sigma_2 = \sigma_0$), or the infinite-frequency limit of the resistor-capacitor mixture, is also analytically tractable, since the action of $\bar{\mathcal{T}}$ now reads:

$$\bar{\mathcal{T}} \qquad \begin{matrix} p \mapsto T(p) \\ \sigma_0 \mapsto 2^{D-2} f_\infty(p) \sigma_0 \end{matrix} \tag{3.17}$$

where

$$f_\infty(p) = 1 + p \tag{3.18}$$

in any dimension.

The macroscopic conductivity therefore reads:

$$\Sigma(\infty, \sigma_0, p) = \sigma_0 \prod_{n \geqslant 0} f_\infty(T^n(p)). \tag{3.19}$$

This expression is infinite for $p \geqslant p_c$, reaches the value σ_0 for $p = 0$ and diverges as $(p_c - p)^{-s}$ for $p \to p_c^-$, where the exponent s is:

$$s = \ln f_\infty(p_c)/\ln \mu. \tag{3.20}$$

In three dimensions, this number reads:

$$s = 0.439\,675 \qquad (D = 3). \tag{3.21}$$

In two dimensions, our approximate scheme $\bar{\mathcal{T}}$ preserves the *self-duality* of the genuine renormalisation group transform $\mathcal{T}$. In particular, we have the well known duality relation

$$\Sigma(\sigma_0, 0, p)\Sigma(\infty, \sigma_0, D(p)) = \sigma_0^2 \tag{3.22}$$

where $D(p)$ is as in (2.8), from which one easily verifies the expected relation:

$$s = t \qquad (D = 2). \tag{3.23}$$

3.3. Scaling functions

The low-frequency conductivity of our resistor-insulator mixture has the following scaling behaviour (see (1.1))

$$\Sigma = \sigma_0 |p - p_c|^t \varphi_\pm[(i\omega/\omega_0)|p - p_c|^{-s-t}] \tag{3.24}$$

when $p \to p_c^\pm$ and $\omega/\omega_0 \to 0$ simultaneously. Although it is straightforward to become convinced that our renormalisation scheme $\bar{\mathcal{T}}$ indeed leads to (3.24) (with negligible oscillations in the variable $\ln|p - p_c|/\ln \mu$), the mathematical study of a function like Σ involves the essentially unexplored area of iterations of analytic functions in *several* complex variables.

Consider first the region $p > p_c$. The scaling function φ_+ is real analytic around the origin:

$$\varphi_+(x) = A_+ + B_+ x + \dots \tag{3.25}$$

where A_+ and B_+ are the amplitudes of the DC (static) conductivity and dielectric constant:

$$\begin{aligned} \Sigma_{DC} &\sim \sigma_0 A_+ (p - p_c)^t \\ \varepsilon &\sim C_0 B_+ (p - p_c)^{-s}. \end{aligned} \tag{3.26}$$

When its argument is large, φ_+ is singular:

$$\varphi_+(x) \underset{\substack{|x|\to\infty \\ |\mathrm{Arg}\, x|<\pi}}{=} K x^u + \dots \tag{3.27}$$

where K and u are as in equations (1.2)-(1.3). The asymptotic expansion (3.27) is indeed valid in a cut plane, since Σ can be singular only for real negative values of Δ of equation (3.9*e*), i.e. for real negative values of σ_2/σ_1. The singularities of Σ have also been studied on a deterministic fractal model of percolation (Clerc *et al* 1985): these authors find that Σ is discontinuous on a Cantor set of the negative real axis in the σ_2/σ_1 variable.

In the non-percolating phase ($p < p_c$), the scaling function $\varphi_-(x)$ is regularly vanishing at the origin:

$$\varphi_-(x) = B_- x + \dots \tag{3.28}$$

where B_- is the amplitude of the conductivity of a dilute superconductor:

$$\Sigma_{super} \sim \sigma_0 B_- (p_c - p)^{-s} \tag{3.29}$$

and of the static dielectric constant:

$$\varepsilon \sim C_0 B_- (p_c - p)^{-s}. \tag{3.30}$$

When its argument is large, φ_- is asymptotic to φ_+, such that Σ has the following well defined behaviour *at* $p = p_c$:

$$\Sigma = K\sigma_0 \exp(\tfrac{1}{2} i \pi u)(\omega/\omega_0)^u \tag{3.31}$$

for $\omega \ll \omega_0$.

3.4. The loss angle

It has been pointed out that the loss angle δ defined through:

$$\tan \delta = \mathrm{Re}\, \Sigma / \mathrm{Im}\, \Sigma \tag{3.32}$$

is of some theoretical and experimental interest (Laugier 1982, Clerc *et al* 1984, 1985).

From the last subsection, since every constant in equation (3.31) is real, we deduce that δ takes the following *universal* value:

$$\delta_c = \tfrac{1}{2}\pi(1 - u) = \tfrac{1}{2}\pi s/(s + t) \tag{3.33}$$

for $p = p_c$ and $\omega \ll \omega_0$.

For $p > p_c$, we have a crossover at $\omega^* \sim \omega_0 (p - p_c)^{s+t}$ such that

$$\begin{aligned} &\tan \delta \sim \omega^{-1} && (\omega \ll \omega^*) \\ &\tan \delta = \tan \delta_c && (\omega^* \ll \omega \ll \omega_0). \end{aligned} \tag{3.34a}$$

For $p < p_c$, $\omega^* \sim \omega_0 (p_c - p)^{s+t}$ and

$$\begin{aligned} &\tan \delta \sim \omega && (\omega \ll \omega^*) \\ &\tan \delta = \tan \delta_c && (\omega^* \ll \omega \ll \omega_0). \end{aligned} \tag{3.34b}$$

What happens at $\omega \gg \omega_0$ and p close to p_c depends upon a non-universal and unexpected criterion, namely the position of p_c with respect to $\frac{1}{2}$. Indeed, if $p_c > \frac{1}{2}$, then $\omega \to \infty$ corresponds to a superconductor problem at $p = 1 - p_c < p_c$, and hence Σ is dominated by resistors and $\tan \delta \sim \omega$ for analyticity reasons. Conversely, if $p_c < \frac{1}{2}$, the equivalent superconductors do percolate at $p = 1 - p_c > p_c$, and $\tan \delta \sim \omega^{-1}$. In other words, whenever $p_c > \frac{1}{2}$, a conducting infinite cluster and an insulating one cannot coexist.

Figures 4 and 5 show log-log plots of $\tan \delta$ against (ω/ω_0) in the 2D and 3D models respectively. The universal δ_c are such that:

$$\begin{aligned} &\tan \delta_c = 1 \text{ (exact)} && (D = 2) \\ &\tan \delta_c = 0.263\,332 && (D = 3). \end{aligned} \tag{3.35}$$

The expected asymptotic regimes are clearly visible. Let us recall that we have: $p_c(D = 2) > \frac{1}{2} > p_c(D = 3)$.

For percolation on a 2D *square* lattice ($p_c = \frac{1}{2}$), two crossovers are expected at $\omega^* \sim \omega_0 |p - p_c|^{s+t}$ and $\omega^{**} \sim \omega_0 |p - p_c|^{-s-t}$, since equation (3.34) still holds and the duality relation now implies:

$$\tan \delta(\omega/\omega_0) \tan \delta(\omega_0/\omega) = 1. \tag{3.36}$$

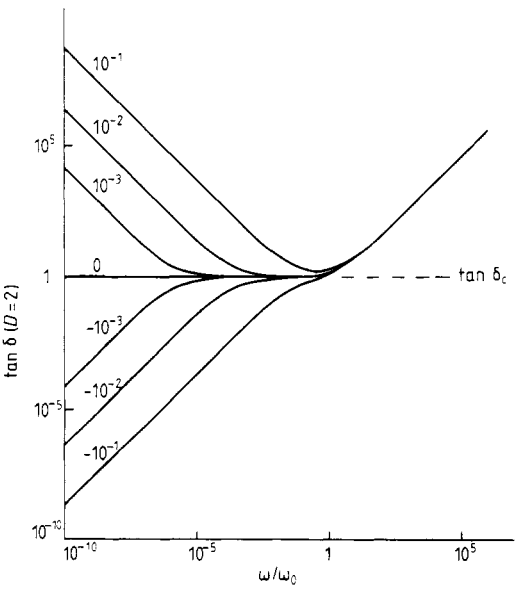

Figure 4. Log-log plot of $\tan \delta$ against (ω/ω_0) in the 2D model. Values of $p - p_c$ are indicated on the curves.

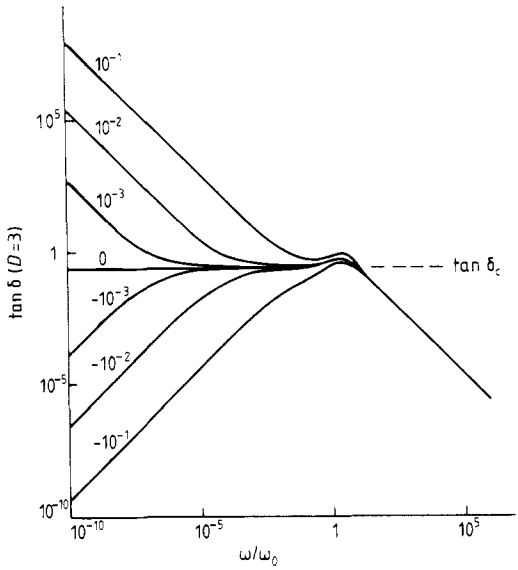

Figure 5. Same as figure 4, for the 3D model.

4. Scaling properties of noise spectra

In this section, we study the properties of the observable noise spectrum of a macroscopic sample of resistor-capacitor mixture.

Two types of noise can be considered: thermal or Nyquist noise, which is independent of the intensity across the sample and related by the Nyquist theorem to the real part of its impedance, and flicker (or $1/f$) noise, which is proportional to the squared intensity (see Dutta and Horn 1981 for a review). In the following, we model the latter by microscopic fluctuations of resistance.

We have chosen to model flicker noise as follows. Each resistor has a small time-dependent dimensionless random component $\Delta(t) \ll 1$: $R = R_0[1+\Delta(t)]$. The noise generators are defined through their common stationary spectral density $s_0(\omega)$:

$$s_0(\omega) = \int \mathrm{d}t \exp(\mathrm{i}\omega t)\langle\Delta(t)\Delta(0)\rangle. \tag{4.1}$$

The quantity we plan to compute is the noise *amplification* $G(p, \omega)$, defined as follows. If $s_{\mathrm{mac}}(\omega)$ is the dimensionless spectrum of a macroscopic sample, let

$$G(p, \omega) = \frac{s_{\mathrm{mac}}(\omega)}{s_0(\omega)} L^D. \tag{4.2}$$

This formulation of resistor noise has been recently used by Clerc *et al* (1985). The factor L^D is such that $G = 1$ is a pure medium (cf. equation (3.11)).

The crucial point in the determination of the renormalisation group transform $\bar{\mathcal{T}}$ in § 3 was: hierarchical lattices only involve sums of two impedances and sums of two conductances. It is therefore sufficient to study the composition of the noise spectra of two sources in series and in parallel.

Consider first two conductors a and b in *series*. Let Z_a, Z_b be their impedances; $\tilde{\Delta}_a(\omega)$, $\tilde{\Delta}_b(\omega)$ the Fourier components of their noise signals; s_a, s_b the associated noise

spectra, defined according to equation (4.1). It is elementary to show that the impedance of the set of conductors in series reads:

$$Z = Z_a + Z_b + (Z_a\tilde{\Delta}_a + Z_b\tilde{\Delta}_b) \tag{4.3}$$

and therefore the composed noise spectrum is

$$s_{\text{ser}} = \frac{|Z_a|^2 s_a + |Z_b|^2 s_b}{|Z_a + Z_b|^2}. \tag{4.4}$$

Consider now the previous two conductors associated in *parallel*. Let $\sigma_a = Z_a^{-1}$, $\sigma_b = Z_b^{-1}$ be their conductances. The conductance of the whole is now:

$$\sigma = \sigma_a + \sigma_b - (\sigma_a\tilde{\Delta}_a + \sigma_b\tilde{\Delta}_b) \tag{4.5}$$

from which we get the composed noise spectrum:

$$s_{\text{par}} = \frac{|\sigma_a|^2 s_a + |\sigma_b|^2 s_b}{|\sigma_a + \sigma_b|^2}. \tag{4.6}$$

Let us now use these basic results in order to extend the transform $\bar{\mathcal{T}}$ of the previous section to noise amplification. Assume the initial distribution of bond impedances is as in equation (3.4), and the corresponding noise spectra are s_1 and s_2 (in the case of the resistor-capacitor mixture at $\omega < \omega_0$, $Z_1 = \sigma_0^{-1}$, $Z_2 = (\mathrm{i}C_0\omega)^{-1}$; $s_1 = s_0$, $s_2 = 0$). The distribution of branch impedances is therefore given by equations (3.6)-(3.9). In order to preserve the coherence of our renormalisation scheme, we impose that each impedance Z_1' (respectively Z_2') is associated to a noise spectrum s_1' (respectively s_2'). The values of s_1', s_2' are chosen through the same criterion as the value of p' (equation (3.8)) in § 3: in the $\omega \to 0$ limit ($|Z_1| \ll |Z_2|$), a branch resistance is Z_1' if its two constituents have resistance Z_1 (implying $s_1' = 2s_1$), and Z_2' in all other cases. The most reasonable way of getting a value of s_2' out of this distribution is to average it. We obtain:

series

$$s_1' = \tfrac{1}{2}s_1 \tag{4.7a}$$

$$s_2' = \frac{4p[|Z_1|^2 s_1 + |Z_2|^2 s_2] + (1-p')|Z_1 + Z_2|^2 s_2}{2(1+p)|Z_1 + Z_2|^2}. \tag{4.7b}$$

Just as we did for equation (3.8), we shall use equation (4.7) whenever $|Z_1| < |Z_2|$. This lengthy discussion can be repeated for noise composition of two branches in parallel. The result is ($|\sigma_1| < |\sigma_2|$):

parallel

$$s_1' = \tfrac{1}{2}s_1 \tag{4.8a}$$

$$s_2' = \frac{4p[|\sigma_1|^2 s_1 + |\sigma_2|^2 s_2] + (1-p)|\sigma_1 + \sigma_2|^2 s_2}{2(1+p)|\sigma_1 + \sigma_2|^2}. \tag{4.8b}$$

$\bar{\mathcal{T}}$ therefore acts as follows: apply equation (4.7) once and then equation (4.8) once in the 2D case; apply equation (4.7) once and then equation (4.8) twice in the 3D case.

Just as in the previous section, the $\omega = 0$ (DC) limit is analytically tractable, since the action of $\bar{\mathcal{T}}$ simplifies:

$\bar{\mathcal{T}}$

$$s_1 \mapsto 2^{-D} g(p) s_1 \tag{4.9a}$$

$$s_2 \mapsto 2^{-D} h(p) s_2 \tag{4.9b}$$

with

$$g(p)=\frac{4-3p^2}{2-p^2} \qquad (D=2) \tag{4.10a}$$

$$g(p)=\frac{4-3p^2}{2-p^2}\,\frac{3(1-p^2)^2+1}{(1-p^2)^2+1} \qquad (D=3) \tag{4.10b}$$

$$\frac{3p+1}{p+1} \qquad (\text{all } D). \tag{4.10c}$$

Consider first the percolating phase ($p>p_c$). Starting from $s_1^{(0)}=s_0$, $s_2^{(0)}=0$ and applying $\bar{\mathcal{T}}$ a large number of times, we get the following limiting result:

$$\lim_{n\to\infty}[(s_1^{(n)}/s_0)2^{Dn}]=G(p,0)=\prod_{n\geqslant 0} g(T^n(p)) \tag{4.11}$$

where G has been defined in equation (4.2). $G(p,0)$ goes to one as p goes to one: this justifies the factor L^D in the definition (4.2) of G. More interesting is the critical regime: as $p\to p_c^+$, $G(p,0)$ diverges according to

$$G(p,0)\underset{p\to p_c^+}{\sim}(p-p_c)^{-X} \tag{4.12}$$

where the critical exponent X reads:

$$X=\ln g(p_c)/\ln \mu \tag{4.13}$$

i.e. numerically

$$X=1.338\,958 \qquad (D=2) \tag{4.14a}$$

$$X=2.343\,155 \qquad (D=3). \tag{4.14b}$$

Since $G(p,\omega)$ is by definition the amplification of the spectrum of the dimensionless quantity $\Delta(t)$, the resistance fluctuations of a macroscopic sample have the following spectral density: $L^{-D}s_0(\omega)G(p,\omega)|Z(\omega)|^2$, where $Z(\omega)$ is the total impedance of the sample.

In the non-percolating phase ($p<p_c$), the static (DC) amplification $G(p,0)$ vanishes identically, since the initial condition $s_2(\omega)=0$, which is valid for all ω, is preserved by the action of $\bar{\mathcal{T}}$ in the $\omega=0$ limit.

At non-zero frequency, $G(p,\omega)$ is non-zero in both phases. For $\omega\ll\omega_0$ and $p<p_c$, $G(p,\omega)$ behaves like

$$G(p,\omega)=l(p)\omega^2/\omega_0^2+\ldots \tag{4.15}$$

where the amplitude $l(p)$ vanishes regularly at $p\to 0$, and diverges as $p\to p_c$ according to

$$l(p)\sim(p_c-p)^{-X-2(s+t)}. \tag{4.16}$$

In the general AC case ($\omega\neq 0$), our procedure leads to a well behaved function $G(p,\omega)$. This amplification exhibits the following scaling behaviour for $p\to p_c^{\pm}$ and $\omega/\omega_0\to 0$ simultaneously:

$$G(p,\omega)=|p-p_c|^{-X}\chi_{\pm}[(\omega/\omega_0)|p-p_c|^{-s-t}]. \tag{4.17}$$

Let us point out a basic difference between the scaling functions (4.17) of noise amplification and those (3.24) of the conductivity. Since the step (4.8) of the action

of $\bar{\mathcal{T}}$ on noise amplification is non-analytic, $\chi_\pm$ do not have any particular analyticity property, and their argument x is to be viewed as a *real* variable. When the variable x gets large, the functions $\chi_\pm$ both behave like:

$$\chi_\pm(x) \sim Qx^{-Y} \tag{4.18}$$

with

$$Y = X/(s+t) \tag{4.19}$$

and where Q is some constant.

In particular, noise amplification *at* the percolation threshold has the following low-frequency behaviour:

$$G(p_c, \omega) \underset{\omega \to 0}{\sim} Q(\omega/\omega_0)^{-Y}. \tag{4.20}$$

The numerical values of this new exponent read:

$$Y = 0.589\,704 \qquad (D=2) \tag{4.21a}$$

$$Y = 0.873\,583 \qquad (D=3). \tag{4.21b}$$

Figures 6 and 7 show log-log plots of $G(p, \omega)$ against (ω/ω_0) for different values of $p - p_c$ in the 2D and 3D models respectively. We present only results for $p > p_c$ for clarity.

The high-frequency behaviour of $G(p, \omega)$ is determined by the same criterion as the one of the loss angle (see § 3.4). Namely, when $p_c > \frac{1}{2}$ (2D case), resistors dominate the $\omega \to \infty$ limit and $G(p, \infty)$ is finite; when $p_c < \frac{1}{2}$ (3D case), capacitors dominate the $\omega \to \infty$ limit, and $G(p, \omega) \sim \omega^{-2}$. This difference is clearly visible in figures 6-7.

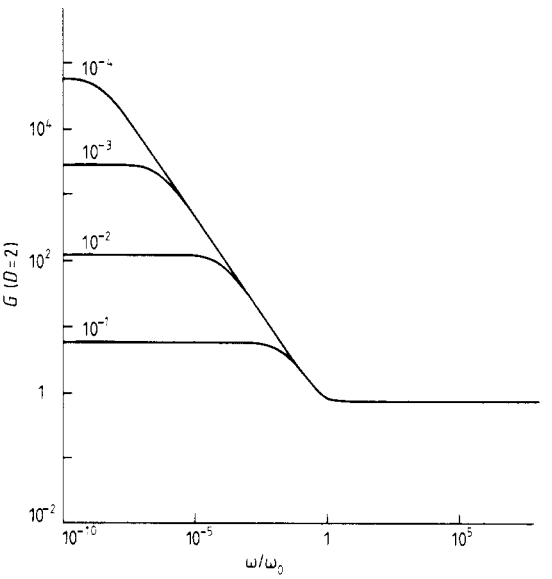

Figure 6. Log-log plot of flicker noise amplification function $G(p, \omega)$ against (ω/ω_0) in the 2D model for $p > p_c$. Values of $p - p_c$ are indicated on the curves.

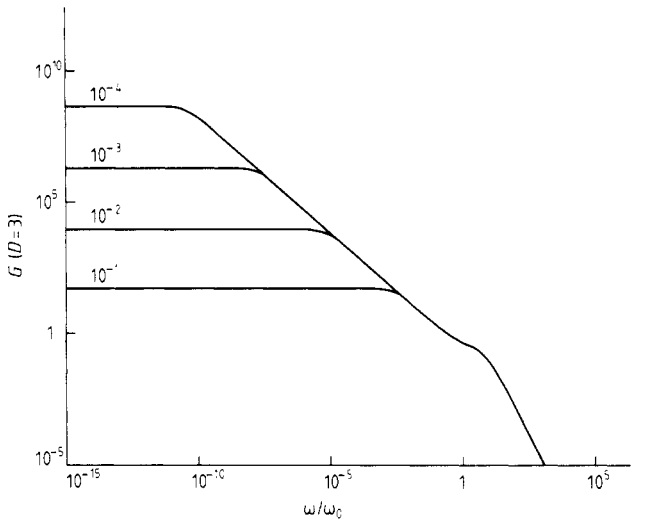

Figure 7. Same as figure 6, for the 3D model.

5. Conclusions

We have used hierarchical lattices in dimension $D=2$ and $D=3$ to study electrical properties of percolation clusters at and around p_c. The static quantities of these models can be computed exactly. The results coincide with those of the Migdal-Kadanoff renormalisation scheme on regular (square and cubic) lattices.

An approximate renormalisation transformation $\bar{\mathcal{T}}$ for the bond conductance distribution leads to a complete determination of the conductivity Σ as a function of p and ω. $\bar{\mathcal{T}}$ has the advantage of preserving the self-duality of the 2D model. Of course the values of s and t are not equal to their (numerically known) values for regular lattices, but the disagreement is never violent: in the worst case, $s(D=3)=0.439\,675$ from equation (3.21), to be compared with 0.75 ± 0.04 from Herrmann *et al* (1984) ($s/\nu=0.85\pm0.04$), Heermann and Stauffer (1981) ($\nu=0.88\pm0.01$).

If we insert (5.2) into (5.1), we get a nonlinear (convolution) integral equation for hierarchical lattices. The critical exponents t and s associated with the exact integral equation (3.2) can be numerically determined by the following procedure, originated by Stinchcombe and Watson (1976). These authors have considered the 2D case, and looked for a fixed probability distribution of the form:

$$p^*(\sigma)=(1-p_c)\delta(\sigma)+p_cR(\sigma) \tag{5.1}$$

which transforms into itself up to a scale λ under the renormalisation operator $\mathcal{T}$:

$$\mathcal{T}\{p^*(\sigma)\}=\lambda p^*(\lambda\sigma). \tag{5.2}$$

If we insert (5.2) into (5.1), we get a nonlinear (convolution) integral equation for the function $R(\sigma)$, which is easy to solve by a discretisation of the variable σ. The exponent $s=t$ is then given by:

$$s=t=\ln\lambda/\ln\mu \tag{5.3}$$

where μ is as in (2.2).

We have generalised this method to the 3D case, where the lack of a duality symmetry forces us to look for *two* fixed distributions:

$$p_1^*(\sigma) = (1-p_c)\delta(\sigma) + p_c R_1(\sigma) \tag{5.4a}$$

$$p_2^*(\sigma) = p_c \delta(\sigma^{-1}) + (1-p_c) R_2(\sigma). \tag{5.4b}$$

Let λ_1, λ_2 denote the associated 'eigenscales' of the operator $\mathcal{T}$. The exponents s and t then read:

$$t = \ln(2\lambda_1)/\ln \mu \tag{5.5a}$$

$$s = -\ln(2\lambda_2)/\ln \mu \tag{5.5b}$$

where the factor 2 occurring in these equations takes into account the geometrical factor L^{D-2} of equation (3.11). Our results read: $\lambda = 1.760$ for $D = 2$ (Stinchcombe and Watson get $\lambda = 1.756$); $\lambda_1 = 1.877$ and $\lambda_2 = 0.351$ for $D = 3$. The associated values of the exponents read:

$$t(D=2) = s(D=2) = 1.334 \tag{5.6a}$$

$$t(D=3) = 2.342 \tag{5.6b}$$

$$s(D=3) = 0.627. \tag{5.6c}$$

However the numerical treatment of the exact integral operator $\mathcal{T}$ is very hard to extend to more complicated quantities, like the AC conductivity or noise amplification. We have therefore used the truncated transform $\bar{\mathcal{T}}$ to study the ω dependence of the conductivity Σ, and consequently of the loss angle δ. The global picture of these curves seems to be very close to experimental ones (see Clerc *et al* 1984, Laugier 1982).

An extension of the mapping $\bar{\mathcal{T}}$ to the resistance noise spectra allows for a detailed study of their critical properties. For $p > p_c$, DC amplification rates diverge with the exponent X: $G_{\omega=0} \sim (p-p_c)^{-X}$. At $p = p_c$, the low-frequency amplification diverges with the exponent Y: $G(p_c, \omega) \sim H(p_c, \omega) \sim (\omega/\omega_0)^{-Y}$. The two critical exponents X and Y are related to the exponent b of Rammal *et al* (1985) through:

$$(s+t)Y = X = (D-b)\nu. \tag{5.7}$$

For $p < p_c$, flicker noise amplification vanishes in the DC limit and behaves like ω^2 for $\omega \ll \omega_0$.

Our results imply the following values for the exponent b:

$$b = 1.181\,204 \qquad (D=2) \tag{5.8a}$$

$$b = 1.090\,977 \qquad (D=3). \tag{5.8b}$$

Table 1 summarises the values of p_c and the critical exponents predicted by our model (truncated transform $\bar{\mathcal{T}}$), together with other predictions: numerical work on regular lattices, and our results (5.6) using Stinchcombe and Watson's method. It would be interesting to have numerical estimates of the noise exponents X, Y, b for regular 2D and 3D lattices.

Since the exponent $Y = 0.873$ is close to unity in the 3D case, the critical amplification by a percolative structure of a microscopic *white* noise roughly leads to a $1/f$ spectrum. This could therefore be an explanation of the occurrence of $1/f$ noise in some disordered materials.

Table 1. Summary of the values of p_c and the critical exponents predicted by our model, compared with other sources: *, our results using Stinchcombe and Watson's method; [a] den Nijs (1979), [b] Herrmann *et al* (1984); [c] Heermann and Stauffer (1981); [d] Derrida *et al* (1983); [e] Wilke (1983).

Quantity	Present model	Other sources
$D=2$		
p_c	0.618 034	$\frac{1}{2}$ (exact)
ν	1.635 279	$\frac{4}{3}$ [a]
$s=t$	1.135 279	1.303 ± 0.01[b]; 1.334*
X	1.338 958	
Y	0.589 704	
b	1.181 204	
$D=3$		
p_c	0.281 837	0.2492 ± 0.0002[c]
ν	1.227 411	0.88 ± 0.01[c]
s	0.439 675	0.75 ± 0.04[b]; 0.627*
t	2.242 559	1.9 ± 0.1[d]; 2.342*
X	2.343 155	
Y	0.873 583	
b	1.090 977	

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